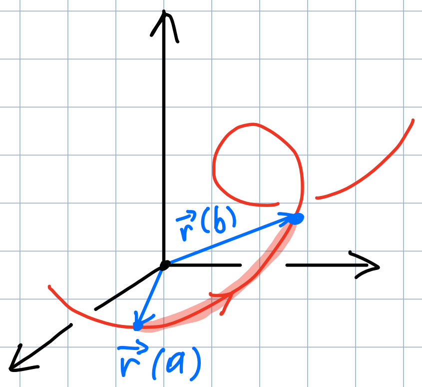
Last time:

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt$$

length of the curve

$$= \int_a^b |\vec{r}\,'(t)|\, dt$$



$C : \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

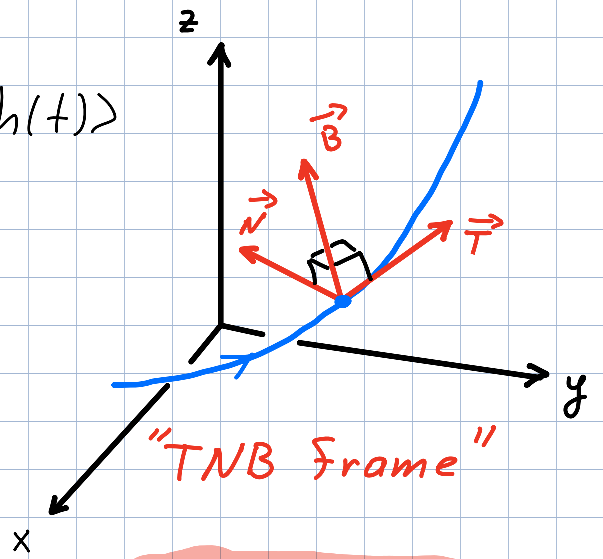
unit

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$$

normal

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

binormal



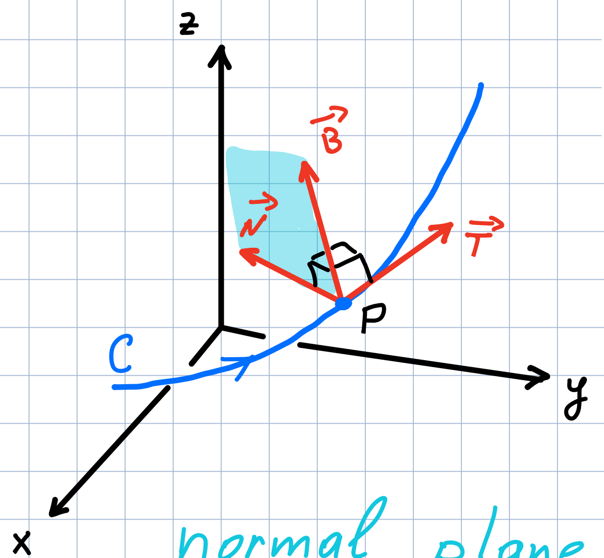
"TNB frame"

Rmk: $T \times N = B$

$T = N \times B$

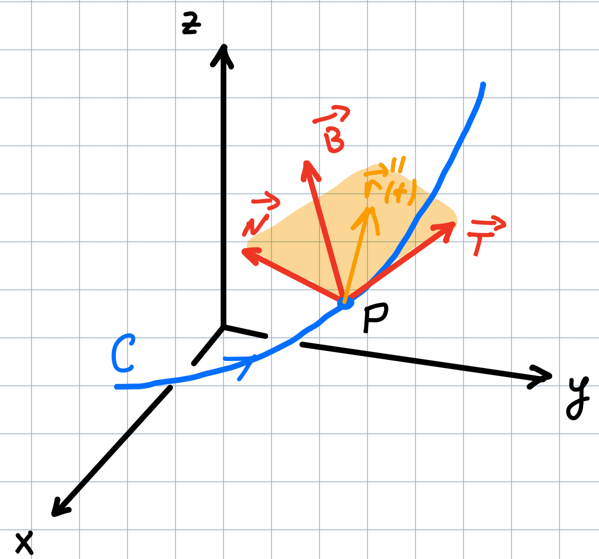
"$\times T = N = B \times$"

$N = B \times T$



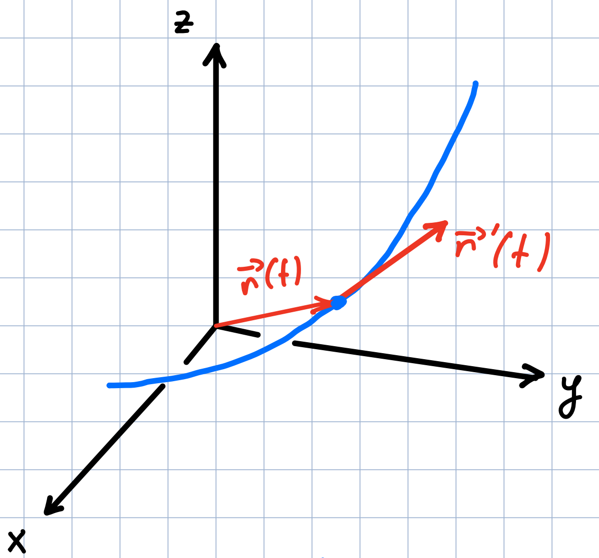
normal plane

osculating plane



Rmk: $\vec{r}\,''(t)$ is in the osculating plane.

Motion in space



particle moving through space;

position at time t: $\vec r(t)$

$$\vec r'(t) = \lim_{h\to 0} \underbrace{\frac{\vec r(t+h)-\vec r(t)}{h}}_{\text{displacement vector per unit time}} = \vec v(t)$$

↑ velocity vector

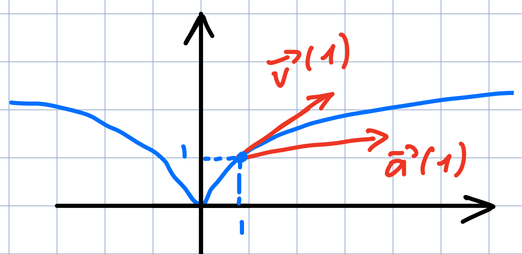
speed: $|\vec v(t)| = |\vec r'(t)| = \frac{ds}{dt}$ ← arc length fct.

↑ rate of change of dist. w.r.t. time

Reminder: $s(t) = \int_a^t |r'(u)|\,du \Rightarrow \frac{ds}{dt} = |r'(t)|$

acceleration: $\vec a(t) = \vec v'(t) = \vec r''(t)$

Ex: On $\mathbb{R}^2$: $\vec r(t) = t^3\,\vec i + t^2\,\vec j$ Find $\vec v(t)$, $\vec a(t)$ and $|\vec v(t)|$ at $t=1$.



Sol:

$\vec v(t) = \vec r'(t) = 3t^2\,\vec i + 2t\,\vec j$

$\vec a(t) = \vec v'(t) = 6t\,\vec i + 2\,\vec j$

$|\vec v(t)| = \sqrt{9t^4 + 4t^2}$

at $t=1$:

$\vec v(1) = \langle 3, 2\rangle$

$\vec a(1) = \langle 6, 2\rangle$

$|\vec v(1)| = \sqrt{13}$

Ex: Moving particle starts at $\vec{r}(0)=\langle 1,0,0\rangle$ with initial velocity $\vec{v}(0)=\vec{i}-\vec{j}+\vec{k}$, and $\vec{a}(t)=4t\,\vec{i}+6t\,\vec{j}+\vec{k}$. Find $\vec{r}(t)$ and $\vec{v}(t)$.

Sol: 1) $\vec{a}(t)=\vec{v}'(t) \Rightarrow \vec{v}(t)=\int \vec{a}(t)\,dt = 2t^2\vec{i}+3t^2\vec{j}+t\vec{k}+\vec{C}$

at $t=0$ $\quad \vec{v}(0)=\vec{i}-\vec{j}+\vec{k}=\vec{C}$

$$\Rightarrow \boxed{\vec{v}(t)=(2t^2+1)\vec{i}+(3t^2-1)\vec{j}+(t+1)\vec{k}}$$

2) $\vec{v}(t)=\vec{r}'(t) \Rightarrow \vec{r}(t)=\int \vec{v}(t)\,dt=\left(\frac{2}{3}t^3+t\right)\vec{i}+(t^3-t)\vec{j}+\left(\frac{1}{2}t^2+t\right)\vec{k}+\vec{D}$

at $t=0$ $\quad \vec{r}(0)=\vec{i}=\vec{D}$

$$\Rightarrow \boxed{\vec{r}(t)=\left(\frac{2}{3}t^3+t+1\right)\vec{i}+(t^3-t)\vec{j}+\left(\frac{1}{2}t^2+t\right)\vec{k}}$$

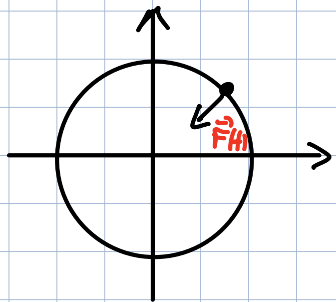
Rmk: In general if we know $\vec{a}(t)$ and $\vec{r}(t_0), \vec{v}(t_0)$ (initial position and velocity) then

$$\vec{v}(t)=\int_{t_0}^{t}\vec{a}(u)\,du+\vec{v}(t_0), \qquad \vec{r}(t)=\int_{t_0}^{t}\vec{v}(u)\,du+\vec{r}(t_0)$$

Rmk: If force $\vec{F}(t)$ acting on particle is known, then we know $\vec{a}(t)$ from $\vec{F}(t)=m\,\vec{a}(t)$ – Newton's 2nd law.

Ex: $\vec{r}(t) = a\cos(wt)\vec{i} + a\sin(wt)\vec{j}$ - object of mass m moves in a circular path with angular speed w. Find $\vec{F}(t)$.

Sol:



$$\vec{v}(t) = \vec{r}\,'(t) = -aw\sin(wt)\vec{i} + aw\cos(wt)\vec{j}$$

$$\vec{a}(t) = \vec{v}\,'(t) = -aw^2\cos(wt)\vec{i} - aw^2\sin(wt)\vec{j}$$

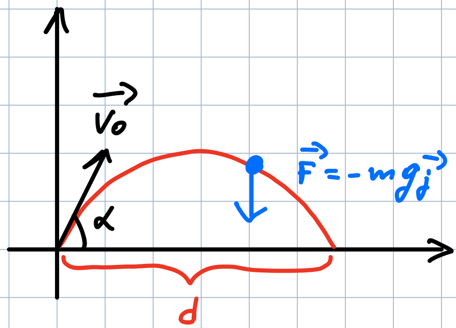
$\Longrightarrow$

$$\vec{F}(t) = m\vec{a}(t) = -mw^2\left(a\cos(wt)\vec{i} + a\sin(wt)\vec{j}\right)$$

$$= -mw^2\,\vec{r}(t)$$

- centripetal force (points toward the origin)

Projectile motion



Projectile is fired at angle of elevation α, initial velocity $\vec{v_0}$. (no air resistance)

Find $\vec{r}(t)$, what value of α maximazes the range d?

Sol:

$\boxed{\vec{a}=\vec{v}\,'}$

1) $\vec{F} = m\vec{a} = -mg\vec{j} \Rightarrow \vec{a} = -g\vec{j}$ (gravity $g = 9.8\ ^{m}/_{s^2}$)

2) $\vec{v}\,'(t) = -g\vec{j} \Rightarrow \vec{v}(t) = \int(-g\vec{j})dt = -gt\vec{j} + \vec{c}$

$v_0 = |\vec{v_0}|$

$\vec{v}(0) = \vec{v_0} = v_0\cos\alpha\,\vec{i} + v_0\sin\alpha\,\vec{j} = \vec{c}$

$\Rightarrow \vec{v}(t) = -gt\vec{j} + \vec{v_0} = v_0\cos\alpha\,\vec{i} + (v_0\sin\alpha - gt)\vec{j}$

$\boxed{\vec{v}=\vec{r}\,'}$

3) $\vec{r}\,'(t) = -gt\vec{j} + \vec{v_0} \Rightarrow \vec{r}(t) = \int(-gt\vec{j} + \vec{v_0})dt = -\frac{1}{2}gt^2\vec{j} + \vec{v_0}t + \vec{D}$

$\vec{r}(0) = 0 = \vec{D} \Rightarrow \vec{r}(t) = -\frac{1}{2}gt^2\vec{j} + \vec{v_0}t$

$= v_0 t\cos\alpha\,\vec{i} + (v_0 t\sin\alpha - \frac{1}{2}gt^2)\vec{j}$

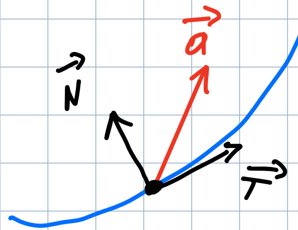
4) param. eq. of trajectory: $x = v_0 t\cos\alpha$

$y = v_0 t\sin\alpha - \frac{1}{2}gt^2$

d = value of x when $y=0 \Rightarrow t(v_0\sin\alpha - \frac{1}{2}gt) = 0 \Rightarrow \left[\begin{array}{l} t=0 \Rightarrow x=0 \\ t=\frac{2v_0\sin\alpha}{g} \end{array}\right.$

$t = \frac{2}{g}v_0\sin\alpha \Rightarrow x = \frac{2}{g}v_0^2\sin 2\alpha \le \frac{2}{g}v_0^2 \Rightarrow d$ is max when $\alpha = \frac{\pi}{4}$.

Tangental and normal components of acceleration:



$$\vec a = \underbrace{a_T}_{\substack{\text{tangental}\\ \text{component}}} \vec T + \underbrace{a_N}_{\substack{\text{normal}\\ \text{component}}} \vec N$$

$$\boxed{a_T} = \vec a \cdot \vec T = \boxed{\frac{\vec r\,'(t) \cdot \vec r\,''(t)}{|\vec r\,'(t)|}}$$

$$\boxed{a_N} = \vec a \cdot \vec N = \boxed{\frac{|\vec r\,'(t) \times \vec r\,''(t)|}{|\vec r\,'(t)|}}$$

Ex: $\vec r(t) = \langle t^2, t^2, t^3 \rangle$. Find a_T, a_N

Sol: $\vec r\,'(t) = \langle 2t, 2t, 3t^2 \rangle$, $|\vec r\,'(t)| = \sqrt{(2t)^2 + (2t)^2 + (3t^2)^2} = \sqrt{8t^2 + 9t^4}$

$\vec r\,''(t) = \langle 2, 2, 6t \rangle$

$$a_T = \frac{\vec r\,'(t) \cdot \vec r\,''(t)}{|\vec r\,'(t)|} = \frac{4t + 4t + 18t^3}{\sqrt{8t^2 + 9t^4}} = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}}$$

$$\vec r\,'(t) \times \vec r\,''(t) = \begin{vmatrix} \vec i & \vec j & \vec k \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} = 6t^2 \vec i - 6t^2 \vec j \quad \Rightarrow \quad a_N = \frac{6t^2\sqrt{2}}{\sqrt{8t^2 + 9t^4}}$$

Tue, Sept 23: **Exam 1**

8:00 – 9:15

Arrive at least 5 min early!

Stepan Center

(Only 10 (best) counts)

Format:

11 multiple choice questions

3 free response questions

Practice exams on Canvas

The exam will cover:

= Chapters 12 and 13

= everything up to and including today's lecture

NO questions on curvature

Lectures 1 – 10

Calculators: NOT allowed